CONSTUM STUDENT COMPETITION PROBLEMS & SOLUTIONS
April 13, 2007

1. *Real-World Application (Agriculture)*: Farmer Bob has $5^{3141592653}$ eggs. He packs them into cartons, each of which holds a dozen eggs, until he no longer has enough eggs to fill a carton. Then he takes these leftover eggs and makes an omelette. How many eggs go into his omelette?

   Solution: We are being asked to reduce a large number modulo 12. Note that $5^2 = 25$, which is congruent to 1 modulo 12. The exponent in the problem is equal to $2n + 1$ for some whole number n, so
   $$5^{3141592653} = 5^{2n+1} = (5^2)^n \cdot 5^1 \equiv 1^n 5^1 = 5 \bmod 12.$$

   Solution 2: (without congruences):

   We are being asked to find the remainder when a large power of 5 is divided by 12. Since the exponent is odd, we may write it as $2n + 1$, where n is a natural number. Now note that
   $$5^{2n+1} = (25)^n \cdot 5 = (24+1)^n \cdot 5 = (1+24+\binom{n}{2}24^2+\cdots+24^n)\cdot 5 = 5+5(24+\binom{n}{2}24^2+\cdots+24^n).$$

   Since the second summand in this last expression is divisible by 12, the remainder upon division by 12 is 5.

2. Fix a positive real number x_0, and define a sequence (x_i) by
   $$x_i = f(x_{i-1}) \qquad \text{for } i > 0,$$
   where $f(x) = \frac{x}{2} + \frac{1}{x}$. Find $\lim_{i\to\infty} x_i$.

   Solution: First, note that the sequence is positive. When does it decrease? increase? This is equivalent to knowing the sign of
   $$g(x) := f(x) - x$$
   for all positive x. Note that g is continuous on the positive real line. Solving $g(x) = 0$ yields $x = \sqrt{2}$. Note that $g(2) < 0$ and $g(1) > 0$. Therefore, for all $x > \sqrt{2}$, $g(x) < 0$ (i.e., $f(x) < x$), and for all $x < \sqrt{2}$, $g(x) > 0$ (i.e., $f(x) > x$).

   Now pick $0 < x < \sqrt{2}$. Numerical experimentation suggests that $f(x) > \sqrt{2}$. Here's how to prove it. Let $h(x) = f(x) - \sqrt{2}$. Since $h'(x) = \frac{1}{2} - \frac{1}{x^2}$, we see that $\sqrt{2}$ is the only positive critical point for h. Since $h(\sqrt{2}) = 0$, and $h(1)$ and $h(2)$ are positive, we see that $h(x)$ is positive for all positive $x \neq \sqrt{2}$.

   Thus, if $x \neq \sqrt{2}$ then $f(x) > \sqrt{2}$.

   Therefore, if we ignore the initial term x_0, we see that our sequence is decreasing, and bounded below by $\sqrt{2}$. Therefore, it has a limit $L \geq \sqrt{2}$. Since g is contiuous near L, we have
   $$g(L) = \lim_{x\to L} g(x) = \lim_{i\to\infty} g(x_i) = 0.$$

   Therefore, $L = \sqrt{2}$.

   I wouldn't be surprised if a simpler proof is possible.

3. Find (with proof) all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\int_x^y f(t)dt = (y - x)f\left(\frac{x + 2y}{3}\right)$$

for all $x, y \in \mathbb{R}$.

Solution: Assume

$$\int_x^y f(t)dt = (y - x)f\left(\frac{x + 2y}{3}\right) \tag{1}$$

for all $x, y \in \mathbb{R}$. Interchanging x, y in (1) we get

$$\int_y^x f(t)dt = (x - y)f\left(\frac{y + 2x}{3}\right) \tag{2}$$

Adding together (1) and (2), we find

$$(y - x)f\left(\frac{x + 2y}{3}\right) + (x - y)f\left(\frac{y + 2x}{3}\right) = 0 \tag{3}$$

for all $x, y \in \mathbb{R}$. From (3) we find

$$f\left(\frac{x + 2y}{3}\right) - f\left(\frac{y + 2x}{3}\right) = 0 \tag{4}$$

for all $x, y \in \mathbb{R}$. Let $u, v \in \mathbb{R}$ be arbitrary numbers. Setting

$$x = \frac{2v - u}{3} \text{ and } y = \frac{2u - v}{3}$$

in (4) will give us $f(u) = f(v)$. Therefore f must be a constant. Conversely, every constant function satisfies (1).

4. Let A be a square matrix with integer entries. Assume that the set consisting of all entries appearing in at least one power of A is bounded. Show that $|\det(A)| \leq 1$.

Solution Since A has integer entries, every power of A will have integer entries, their absolute values being bounded by some constant M. But there are only finitely many square integer matrices of a given size with the property that the absolute values of all of their entries are bounded by M. Consequently, two powers of A must coincide:

$$A^k = A^l, \tag{1}$$

with $1 \leq k < l$. If $\det(A) = \lambda$, by taking determinants in (1) we get $\lambda^k = \lambda^l$ or

$$\lambda^k(1 - \lambda^{l-k}) = 0. \tag{2}$$

Keeping in mind that $\lambda \in \mathbb{Z}$, from (2) it turns out that $\lambda \in \{-1, 0, 1\}$. The conclusion follows.

5. (In memory of Euler)

   The problem of finding the exact value of the infinite sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ was considered by 17th century mathematicians, including Mengoli, Leibniz and Bernoulli. Unable to solve the problem, Jacob Bernoulli wrote:

   If anyone finds and communicate to us that which thus far eluded our efforts, great will be our gratitude.

   about this problem, in 1689. It was Euler, regarded as one of the greatest mathematicians of all times, who first established the remarkable identity

   $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ in 1735. Given Euler's formula, it is much easier to find the exact value of the similar sum $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$. Find it.

   Solution:

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \frac{1}{4}\frac{\pi^2}{6}$$

   Therefore, $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{6} - \frac{1}{4}\frac{\pi^2}{6} = \frac{\pi^2}{8}$

6. Find the limit:

$$\lim_{x\to\infty} \frac{1}{2x^2} \int_0^{x^2} \sqrt{1+e^{-t}}dt$$

   Solution: Since the improper integral $\int_0^{\infty} \sqrt{1+e^{-t}}dt$ is divergent (the integrand is greater than 1 on $(0,\infty)$), we have $0 \cdot \infty$, an indeterminate case. Converting this to the 0 over 0 case, and applying l'hospital's rule (and recalling the Fundamental Theorem of Calculus) we get

$$\lim_{x\to\infty} \frac{\sqrt{1+e^{-x^2}} \cdot 2x}{4x} = \frac{1}{2}$$

7. How many positive integer divisors does 2007^{2007} have?

   Solution: Note the prime factorization of 2007 is $3^2 \cdot 223$. Thus, $2007^{2007} = 3^{4014} \cdot 223^{2007}$ and hence its positive divisors must all be of the form $3^i \cdot 223^j, 0 \leq i \leq 4014, o \leq j \leq 2007$. Thus, it has $4015 \cdot 3008 = 8,062,120$ positive divisors.

8. In celebration of today being "Blame Somebody Else Day," you buy a cube as a gift for Dr. Tom Dence, President of the Ohio Section of the MAA. (If you forgot to buy this, don't worry, you can blame someone else for forgetting.) In turns out that the only wrapping paper that you have has the shape pictured here, where each of the small squares has sides of length 2 inches. Amazingly, this wrapping paper is exactly the proper size to cover the entire surface of your cube without overlaps. (See picture)

   (a) What is the volume of the cube you purchased?

   (b) Draw dotted lines on this picture to show where you will have to fold the wrapping paper in order to cover the cube.

   Solution: Note the area of the wrapping paper is $12 \cdot 4 = 48$ in^2. Thus, the surface area of the cube is 48in^2. Hence, $6s^2 = 48$ and so $s = \sqrt{8}$. Thus, the volume of the cube is $s^3 = 16\sqrt{2}$in^3. The following picture displays how to fold the wrapping paper:

9. Suppose that there was a cancer diagnostic test that was 95% accurate both on those who do and those who do not have the disease. If 0.4% of the population have cancer, compute the probability that a tested person has cancer given that his or her test results indicate so.

   Solution Let C be the event that a person has cancer and let T be the event that a person tests positive for cancer. So we have that $P(T|C) = 0.95$ and $P(T'|C') = 0.95$. Also, $P(C) = 0.004$. We want to find $P(C|T)$. So,

$$\begin{aligned} P(C|T) &= \frac{P(C \cap T)}{P(T)} = \frac{P(T|C)P(C)}{P(T \cap C) + P(T \cap C')} \\ &= \frac{P(T|C)P(C)}{P(T|C)P(C) + P(T|C')P(C')} \\ &= \frac{(0.95)(0.004)}{(0.95)(0.004) + (0.05)(0.996)} = \frac{0.0038}{0.536} = 0.0709 \end{aligned}$$

10. The Fibonacci sequence, $1, 2, 3, 5, 8, 13, \ldots$ is given by the recursive formula

$$F_{n+1} = F_n + F_{n-1}$$

where $F_1 = 1$ and $F_2 = 2$. Let $a_n = \dfrac{F_n}{F_{n-1}}$.

(a) Divide the above equation by F_n to find an equation relating a_{n+1} to a_n.

(b) Show that $\frac{3}{2} \le a_n \le 2 \ \forall n \ge 2$.

(c) For each $n \ge 3$, prove that

$$|a_{n+1} - a_n| \le \left(\frac{2}{3}\right)^2 |a_n - a_{n-1}|.$$

(d) Show that (a_n) is a Cauchy sequence and therefore converges to a limit.

(e) What is the limit?

Solution:

(a) $a_{n+1} = 1 + \dfrac{1}{a_n}$

(b) Note that $a_2 = 2$ and $a_3 = 3/2$. So, $3/2 \le a_2 \le 2$ and $3/2 \le a_3 \le 2$. Suppose that $3/2 \le a_k \le 2$. Then, $a_k \le 2 \Rightarrow 1/2 \le 1/a_k \Rightarrow 3/2 \le 1 + 1/a_k \Rightarrow 3/2 \le a_{k+1}$ and $3/2 \le a_k \Rightarrow 1/a_k \le 2/3 \Rightarrow 1 + 1/a_k \le 5/3 \Rightarrow a_{k+1} \le 2$. By induction we have that $3/2 \le a_n \le 2$ for $n \ge 2$.

(c) $|a_4 - a_3| = |1 + 1/a_3 - (1 + 1/a_2)| = |1/a_3 - 1/a_2| = \dfrac{|a_2 - a_3|}{|a_2||a_3|} \le \left(\dfrac{2}{3}\right)^2 |a_3 - a_2|$

$|a_{n+1} - a_n| = |1 + 1/a_n - (1 + 1/a_{n-1})| = |1/a_n - 1/a_{n-1}| = \dfrac{|a_n - a_{n-1}|}{|a_n||a_{n-1}|} \le \left(\dfrac{2}{3}\right)^2 |a_{n-1} - a_n|$ for $n \ge 3$.

(d) First note that $|a_{n+1} - a_n| \le \left(\frac{2}{3}\right)^2 |a_{n-1} - a_n| \le \left(\frac{2}{3}\right)^4 |a_{n-2} - a_{n-1}| \le \left(\frac{2}{3}\right)^6 |a_{n-3} - a_{n-2}|$. We will show that $|a_{n+1} - a_n| \le \left(\frac{2}{3}\right)^{2(n-2)} |a_3 - a_2|$. Part (c) shows the base case. Suppose that $|a_{k+1} - a_k| \le \left(\frac{2}{3}\right)^{2(k-2)} |a_3 - a_2|$. Then, by part (c) $|a_{k+2} - a_{k+1}| \le \left(\frac{2}{3}\right)^2 |a_{k+1} - a_k|$ and by the inductive hypothesis, we have $|a_{k+2} - a_{k+1}| \le \left(\frac{2}{3}\right)^2 |a_{k+1} - a_k| \le \left(\frac{2}{3}\right)^2 \left(\frac{2}{3}\right)^{2(k-2)} |a_3 - a_2| = \left(\frac{2}{3}\right)^{2(k+1-2)} |a_3 - a_2| = \left(\frac{2}{3}\right)^{2(k-1)} |a_3 - a_2|$ as desired.
To show that (a_n) is Cauchy, let $\epsilon > 0$ be given. Then, by the Archimedian Principle, $\exists N \in \mathbb{N}$ such that $|a_3 - a_2| \frac{729}{80} \left(\frac{4}{9}\right)^N < \epsilon$ Now, $\forall n > m > N$ we have $|a_n - a_m| = |(a_n - a_{n-1}) + (a_{n-1} - a_{n-1}) + \cdots + (a_{m+1} - a_m)| \le |(a_n - a_{n-1})| + |(a_{n-1} - a_{n-1})| + \cdots + |(a_{m+1} - a_m)|$ $\le \left(\frac{2}{3}\right)^{2(n-3)} |a_3 - a_2| + \left(\frac{2}{3}\right)^{2(n-4)} |a_3 - a_2| + \cdots + \left(\frac{2}{3}\right)^{2(m-2)} |a_3 - a_2| =$ $|a_3 - a_2| \displaystyle\sum_{k=m}^{n-1} \left(\frac{2}{3}\right)^{2(k-2)} \le |a_3 - a_2| \sum_{k=m}^{\infty} \left(\frac{2}{3}\right)^{2k-4} = |a_3 - a_2| \frac{81}{16} \sum_{k=m}^{\infty} \left(\frac{4}{9}\right)^k =$ $|a_3 - a_2| \dfrac{81}{16} \left(\dfrac{4}{9}\right)^m \left(\dfrac{9}{5}\right) = |a_3 - a_2| \dfrac{729}{80} \left(\dfrac{4}{9}\right)^m \le |a_3 - a_2| \dfrac{729}{80} \left(\dfrac{4}{9}\right)^N < \epsilon$. Thus the sequence converges.

(e) Suppose $a_n \to L$. Then $L = 1 + \dfrac{1}{L} \Rightarrow L^2 - L - 1 = 0 \Rightarrow L = \dfrac{1 \pm \sqrt{5}}{2} \Rightarrow L = \dfrac{1 + \sqrt{5}}{2}$ since $a_n > 0$.