2015 Leo Schneider Student Team Competition

1. Find $\int_0^\infty \frac{1}{\sqrt{e^x-1}}\,dx$.

2. A fair die and an unfair die are in a bag. The probability of rolling a six with the unfair die is 1/4. One of the dice is randomly drawn from the bag (each one is equally likely to be chosen). A six is rolled with this die. You roll this same die again. What is the probability that a six is rolled?

3. Tom and Jerry decide today that they will not pick on each other on the n-th day if $a_n = -1$ where $a_1 = 1$, $a_2 = 1$, $a_3 = -1$, and a_n for $n > 3$ is inductively given by $a_n = a_{n-1}a_{n-3}$. Will they pick on each other on the 1776-th day? Prove your answer.

4. For each positive integer k, let
$$A_k = \begin{pmatrix} 1 & k & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
   (a) Find a closed form expression for the matrix A_1^n in terms of n. Prove your answer.
   (b) Find all ordered pairs (k, n) of positive integers for which $A_k^n = A_{75}$.

5. Find the limit
$$\lim_{n\to\infty} \left[\frac{(1+\frac{1}{n})^n}{e}\right]^n$$

6. How many solutions does the equation $20x + 15y = 2015$ have over the positive integers?

7. A non-empty set of positive integers is said to be square-valued if the product of all of its elements is a perfect square. How many (non-empty) subsets of $\{10, 22, 30, 42, 55, 231\}$ are square-valued?

8. For the function, $f(x) = \ln\left(1 - \frac{1}{x^2}\right)$, find the value of:
$$f'(2) + f'(3) + f'(4) + \cdots + f'(2015).$$

9. Let S be the set of vertices of a regular 36-gon. What is the smallest value of n for which any subset of S of size n must contain the three vertices of some equilateral triangle? You must prove your answer.

10. Twenty calculus students are comparing grades on their first two quizzes of the year. The class discovers that whenever any pair of students consult with one another, these two students received the same grade on their first quiz or they received the same grade on their second quiz (or both). Prove that the entire class received the same grade on at least one of the two quizzes.