2011 Leo Schneider Student Mathematics Competition

Problem Solutions

March 25, 2011

1. If x and y are non-zero real numbers such that

$$|x| + y = 3 \qquad \text{and} \qquad |x|y + x^3 = 0,$$

then determine the value of $x - y$.

Solution:

- If $x > 0$, then $x + y = 3$ and $y + x^2 = 0$. Eliminate y from these simultaneous equations to obtain $x^2 - x + 3 = 0$, which has no real roots.
  Equivalently, note that the graphs of $y = 3 - x$ and $y = -x^2$ do not intersect.
- If $x < 0$, then we have $-x + y = 3$ and $-y + x^2 = 0$, which have a simultaneous real solution, so $x - y = -(-x + y) = -3$.

2. Suppose that $f(2) = 4$, $f'(2) = 3$, $h(8) = 2$, and $h'(8) = 5$. If $g(x) = h(xf(x))$, find $g'(2)$.

Solution:
By the Chain and Product Rules,

$$g'(x) = h'(xf(x))[f(x) + xf'(x)], \text{ so}$$

$$\begin{aligned} g'(2) &= h'(2f(2))[f(2) + 2f'(2)] \\ &= h'(8)[4 + 2(3)] \\ &= 5 \cdot 10 \\ &= 50. \end{aligned}$$

3. Prove that for all odd integers n,

$$F(n) = n^4 - 4n^2 + 3$$

is divisible by 48.

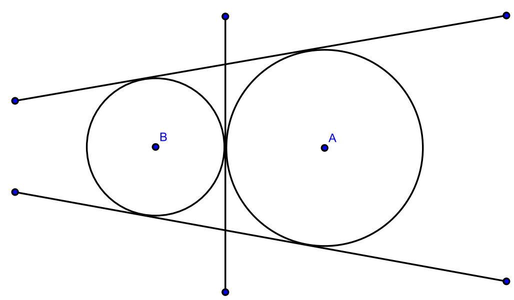
Solution: (n odd $\Rightarrow n = 2k + 1$ for some $k \in \mathbb{Z}$)

$$\begin{aligned} F(n) = n^4 - 4n^2 + 3 \quad &= (n^2 - 3)(n^2 - 1) \\ &= ((2k + 1)^2 - 3)((2k + 1) - 1) \\ &= (4k^2 + 4k + 1 - 3)(4k^2 + 4k + 1 - 1) \\ &= (4k^2 + 4k - 2)(4k^2 + 4k) \\ &= 2(2k^2 + 2k - 1) \cdot 4k(k + 1) \\ &= 8k(k + 1)(2k^2 + 2k - 1) \end{aligned}$$

Since either k or $k + 1$ must be even, we know 16 divides $F(n)$, and it only remains to show that 3 divides $F(n)$. If $k \equiv 0 \bmod 3$, then 3 divides k and hence $F(n)$. If $k \equiv 2 \bmod 3$, then 3 divides $k + 1$ and hence $F(n)$. If $k \equiv 1 \bmod 3$, then 3 divides $2k^2 + 2k - 1$ and hence $F(n)$. In any case, $F(n)$ must also be divisible by 48 as it is divisible by both 16 and 3.

4. Points A and B are 5 units apart. How many lines in a given plane containing A and B are 2 units from A and 3 units from B?

Solution:
The set of lines that are 2 units from the point A is the set of tangents to the circle with center A and radius 2. Similarly, the set of lines that are 3 units from point B is the set of tangents to the circle with center B and radius 3. Thus the desired set of lines is the set of common tangents to the two circles. Since $|AB| = 5 = 2 + 3$, these two circles are tangent externally, so they have three common tangents.



5. Find a function $f(x)$ that satisfies $f(x) = \sqrt{\int_0^x f(t)^2 + f'(t)^2 \; dt + 9}$, for all $x \geq 0$.

Solution:
We begin by squaring both sides and differentiating. After doing so, we get

$$2f(x)f'(x) = f(x)^2 + f'(x)^2.$$

This can be written as

$$(f(x) - f'(x))^2 = 0.$$

This is equivalent to $f'(x) = f(x)$. Hence, $f(x) = Ce^x$, where C can be any constant. Since the definition of $f(x)$ implies that

$$f(0) = \sqrt{\int_0^0 f(t)^2 + f'(t)^2 \; dt + 9} = \sqrt{9} = 3,$$

it follows that $C = 3$ and $f(x) = 3e^x$.

6. When this sum of determinants

$$f(x) = \begin{vmatrix} x^6 & x^5 & x^4 & x^3 \\ x^5 & x^4 & x^3 & x^2 \\ x^4 & x^3 & x^2 & x \\ x^3 & x^2 & x & 1 \end{vmatrix} + \begin{vmatrix} x^4 & x^3 & x^2 \\ x^3 & x^2 & x \\ x^2 & x & 1 \end{vmatrix} + \begin{vmatrix} x^2 \cos x & x \sin x \\ -x^2 \sin x & x \cos x \end{vmatrix}$$

is expanded and simplified, it is a polynomial in terms of x. What is the degree of this polynomial? Justify your answer!

Solution:

$$\begin{vmatrix} x^6 & x^5 & x^4 & x^3 \\ x^5 & x^4 & x^3 & x^2 \\ x^4 & x^3 & x^2 & x \\ x^3 & x^2 & x & 1 \end{vmatrix} = \begin{vmatrix} x^4 & x^3 & x^2 \\ x^3 & x^2 & x \\ x^2 & x & 1 \end{vmatrix} = 0, \text{ and}$$

$$\begin{vmatrix} x^2 \cos x & x \sin x \\ -x^2 \sin x & x \cos x \end{vmatrix} = x^3 \cos^2 x + x^3 \sin^2 x = x^3(\cos^2 x + \sin^2 x) = x^3,$$

so $f(x) = x^3$ is a polynomial of degree 3.

7. Nine chairs in a row are to be occupied by six students and Professors Alpha, Beta and Gamma. These three professors arrive before the six students and decide to choose their chairs so that each professor will be between two students. In how many ways can Professors Alpha, Beta and Gamma choose their chairs?

Solution:
Imagine the six students standing in a row before they are seated. There are 5 spaces between them, each of which may be occupied by at most one of the 3 professors. Therefore, there are ${}_5P_3 = 5 \cdot 4 \cdot 3 = 60$ ways the three professors can select their places.

8. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that $f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n}$ for each $n \geq 1$. Let $\lambda \in [-1, 1]$. Show that there exists a decreasing sequence $(x_n)_{n \geq 1}$ with $\lim_{n \to \infty} x_n = 0$ and $f(x_n) = \lambda x_n$ for each $n \geq 1$.

Solution:
If $\lambda = \pm 1$, then there is nothing to prove (take $x_n = \frac{1}{2n}$ for $\lambda = 1$ and $x_n = \frac{1}{2n-1}$ for $\lambda = -1$). Let $\lambda \in (-1, 1)$. Since $f(x)$ is continuous, the function $g(x) = \frac{f(x)}{x}$ will be continuous on every closed interval $[a, b] \subset (0, \infty)$. For every $n \geq 1$, $g\left(\frac{1}{2n-1}\right) = -1$ and $g\left(\frac{1}{2n}\right) = 1$, so by the Intermediate Value Theorem, we can pick an $x_n \in \left(\frac{1}{2n}, \frac{1}{2n-1}\right)$ such that $g(x_n) = \lambda$. Then $(x_n)_{n \geq 1}$ will be the desired sequence, as it converges decreasingly to zero and $f(x_n) = x_n \cdot g(x_n) = \lambda x_n$ for all $n \geq 1$.

9. (a) Show that the sum

$$1+2+3+\cdots+2010+2011+2010+\cdots+3+2+1$$

is the square of an integer.

(b) Generalize the result in (a), with proof.

Solution:

(a) We may write the sum as the sum of two arithmetic progressions:

$$1+2+3+\cdots+2011=\frac{2011\cdot 2012}{2},$$

and

$$1+2+3+\cdots+2010=\frac{2010\cdot 2011}{2}.$$

Then the sum of the two is

$$\frac{2011(2010+2012)}{2}=2011^2.$$

(b) In general,

$$\begin{aligned}1+2+3+\cdots+(n-1)+n]+[(n-1)+\cdots+3+2+1] &= \frac{n(n+1)}{2}+\frac{(n-1)n}{2}\\ &= \frac{n^2+n+n^2-n}{2}\\ &= \frac{2n^2}{2}\\ &= n^2.\end{aligned}$$

10. An urn contains twenty-one balls, each bearing one of the numbers $\{1,2,3,4,5,6\}$. One ball is numbered 1, two balls bear the number 2, three balls bear number 3, four balls bear number 4, five balls bear number 5, and six balls ball bear number 6. If two balls are drawn at random (without replacement), what is the probability that the sum of the two numbers on them is more than 5?

Solution:
It is easier to calculate the complementary probability (the probability that the sum of the two number is less than or equal to 5). There are $\binom{21}{2}=210$ different pairs of balls. Of these,

$$\binom{1}{1}\binom{2}{1}=1\cdot 2=2 \text{ have sum 3;}$$

$$\binom{1}{1}\binom{3}{1}+\binom{2}{2}=1\cdot 3+1=4 \text{ have sum 4; and}$$

$$\binom{1}{1}\binom{4}{1}+\binom{2}{1}\binom{3}{1}=4+6=10 \text{ have sum 5.}$$

Thus, a total of 16 have a sum that is less than or equal to 5, and the remaining 194 have sum greater than 5. The probability that the sum of the two numbers on them is more than 5 is therefore $194/210=97/105$.