# 2013 Leo Schneider Student Team Competition

1. For the arithmetic sequence $a_1, a_2, ..., a_{16}$, it is known that $a_7 + a_9 = a_{16}$. Find each subsequence of three terms that forms a geometric sequence.

   **Solution:**

   $a_{16} = a_1 + 15d$, where $d$ is the common difference, $a_7 = a_1 + 6d$ and $a_9 = a_1 + 8d$. Since $a_7 + a_9 = a_{16}$, $a_1+6d+a_1+8d = a_1+15d$. Therefore, $d = a_1$. Therefore, the $i$-th term of the sequence $a_i = a_1+(i-1)d$, where $i = 1, 2, 3, ..., 16$. Since $d = a_1$, $a_i = a_1 + ia_1 - a_1 = ia_1$ so that $a_1 = 1a_1$, $a_2 = 2a_1$, $a_3 = 3a_1$,....

   Consequently, one subsequence forming a geometric sequence is $a_1$, $a_2$, $a_4$ (with a common ratio $r$=2). A second subsequence is $a_1$, $a_3$, $a_9$ (with a common ratio $r$=3). A third subsequence is $a_1$, $a_4$, $a_{16}$ (with common ratio $r$=4). The fourth, fifth, and sixth subsequences are $a_2$, $a_4$, $a_8$ ($r$=2); $a_3$, $a_6$, $a_{12}$ ($r$=2); $a_4$, $a_8$, $a_{16}$ ($r$=2);

   For non-integer values of $r$ such that $1 < r < 4$, we have only $r = \frac{3}{2}$ and $r = \frac{4}{3}$, so that additional sequences are $a_4$, $a_6$, $a_9$ and $a_9$, $a_{12}$, $a_{16}$

2. Compute the limit:

   $$\lim_{x\to\infty} \frac{1}{xe^x} \int_{x^2}^{(x+1)^2} e^{\sqrt{t}}\, dt.$$

   **Solution:**

   Applying L'Hospital's Rule, the Fundamental Theorem of Calculus, and the Chain Rule, we get

   $$\begin{aligned}
   \lim_{x\to\infty} \frac{\int_{x^2}^{(x+1)^2} e^{\sqrt{t}}\, dt}{xe^x} &= \lim_{x\to\infty} \frac{\frac{d}{dx}\int_{x^2}^{(x+1)^2} e^{\sqrt{t}}\, dt}{\frac{d}{dx}(xe^x)} \\
   &= \lim_{x\to\infty} \frac{2(x+1)e^{x+1} - 2xe^x}{xe^x + e^x} \\
   &= \lim_{x\to\infty} \frac{2e(x+1) - 2x}{x+1} \\
   &= 2e - 2.
   \end{aligned}$$

3. Part a. Let $f(x) = e^x \sin x$. Find $f^{(10)}(0)$, the 10th derivative of $f$ evaluated at $x = 0$.

   Part b. Let $f(x) = e^x \sin x$. Find $f^{(2013)}(0)$, the 2013th derivative of $f$ evaluated at $x = 0$.

   **Solution to part a.** If $\sum_{n=0}^{\infty} a_n x^n$ is the Maclaurin series for $f$, then $f^{(10)}(0) = 10! \cdot a_{10}$. One way to find $a_{10}$ is to multiply the Maclaurin series for $y = e^x$ and $y = \sin x$ together and keep track of the coefficient of $x^{10}$. In this way, we see that

$$a_{10} = \frac{2}{9!} - \frac{2}{3!7!} + \frac{1}{5!5!}.$$

   Thus, $f^{(10)}(0) = 10! a_{10} = \boxed{32}$.

   **Solution to part b.** It is easy to verify that $f(0) = 0$, $f'(0) = 1$, $f''(0) = 2$, $f'''(0) = 2$, and $f^{(4)}(x) = -4f(x)$. Thus, $f^{(2013)}(0) = (-4)^{2012/4} = -2^{1006}$. (Also, this gives us another way to solve part a: $f^{(10)}(0) = 2(-4)^{8/4} = 32$.)

4. Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers $x$ satisfying $x^4 + 36 \leq 13x^2$.

   **Solution:**
   The condition $x^4 + 36 \leq 13x^2$ is equivalent to

$$x^4 - 13x^2 + 36 = (x^2 - 9)(x^2 - 4) = (x + 3)(x - 3)(x + 2)(x - 2) \leq 0,$$

   which is satisfied only on $[-3, -2]$ and on $[2, 3]$. Since $f'(x) = 3x^2 - 3 = 3(x + 1)(x - 1) > 0$ on $[-3, -2]$ and on $[2, 3]$, the function $f$ is increasing on both intervals. It follows that the maximum value is $\max\{f(-2), f(3)\} = 18$.

5. Let $N$ be a positive integer containing exactly 2013 digits none of whose digits is zero. Show that $N$ is either divisible by 2012 or $N$ can be changed to an integer that is divisible by 2013 by replacing some but not all of its digits by zero.

   **Solution:**
   Let $N = b_1 b_2 \ldots b_{2013}$ be an integer where each digit $b_i > 0$ $(i = 1, 2, \ldots 2013)$. Let $N_0 = 0$, and for any $1 \leq k \leq 2013$, let $N_k$ denote the number obtained by replacing all but the first $k$ digits of $N$ by the digit 0. By the pigeon hole principle, there exists $0 \leq k_1 < k_2 \leq 2013$ such that $N_{k_1}$ and $N_{k_2}$ are congruent modulo 2013. So the difference $N_{k_1} - N_{k_2}$ is divisible by 2013. Since $N_{k_1} - N_{k_1}$ is obtained from $N$ by replacing some but not all of the digits of $N$ by zero, we are done.

6. Prove that $2^{2013}+3$ is a multiple of 11.

**Solution.** Working mod 11, we have that

$$\begin{aligned} 2^{2013}+3 &= 2^3 \cdot 2^{2010}+3 \\ &= 8(2^5)^{402}+3 \\ &= 8(32)^{402}+3 \\ &\equiv 8(-1)^{402}+3 \\ &\equiv 0. \end{aligned}$$

7. A random number generator randomly generates integers from the set $\{1, 2, \ldots, 9\}$ with equal probability. Find the probability (with explanation) that after $n$ numbers are generated, their product is a multiple of 10.

**Solution:**

The product is a multiple of 10 if an only if at least one 5 and at least one even integer have been generated. If $A$ is the event that a 5 has been generated, and $B$ is the event that at least one even integer has been generated, then we are looking for $\Pr(A \cap B)$. Letting $E'$ denote the complement of an event $E$, we know that

$$\begin{aligned} \Pr((A \cap B)') &= \Pr(A' \cup B') \\ &= \Pr(A') + \Pr(B') - \Pr(A' \cap B'). \end{aligned}$$

The event $A' \cap B'$ represents the case in which neither a 5 nor an even integer has been generated, and consequently $\Pr(A' \cap B') = (\frac{4}{9})^n$. Since $\Pr(A') = (\frac{8}{9})^n$ and $\Pr(B') = (\frac{5}{9})^n$, it follows that

$$\Pr((A \cap B)') = \left(\frac{8}{9}\right)^n + \left(\frac{5}{9}\right)^n - \left(\frac{4}{9}\right)^n,$$

and

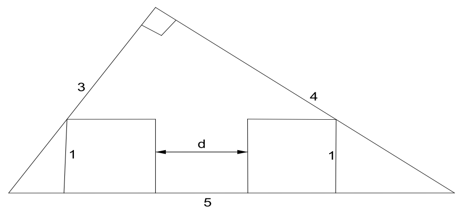
$$\Pr(A \cap B) = 1 - \Pr((A \cap B)') = 1 - \left(\frac{8}{9}\right)^n - \left(\frac{5}{9}\right)^n + \left(\frac{4}{9}\right)^n.$$

8. Planet A is going to launch $n$ missiles at Planet B, which has $n$ cities. Each missile will hit exactly one city. For each missile, the Planet B city that gets hit is completely random. Find the probability that exactly one city on Planet B will not get hit with any of the $n$ missiles.

**Solution:**

The probability is $\frac{\binom{n}{2}n!}{n^n} = \frac{(n-1)(n-1)!}{2n^{n-2}}$. There are many ways to obtain this. Here is one. The denominator is $n^n$ because this is the number of ways to place $n$ missiles in $n$ cities. The numerator is the number of ways of placing the missiles such that exactly one city is not hit. There are $n$ ways to specify the city that does not get hit. There are $n-1$ ways of choosing the city that gets hit with two missiles. There are $\binom{n}{2}$ ways of picking the 2 missiles to hit this city. And there are $(n-2)!$ ways of placing the remaining $n-2$ missiles into the $n-2$ cities, one missile in each city. The product of these is the numerator $n(n-1)\binom{n}{2}(n-2)! = \binom{n}{2}n!$.

9. Two unit squares stand on the hypotenuse of a (3,4,5) triangle in such a way that they line inside the triangle, and a corner of one touches the side of length 3 and a corner of the other touches the side of length 4, as shown in the figure to the right. What is the distance $d$ between the squares?




**Solution:**

Because the small triangle in the lower left is similar to the (3,4,5) triangle, we know that the side of length 3 in the figure has slope $\frac{4}{3}$, so the base of the small triangle in the lower left is $\frac{3}{4}$. The side of length 4 has slope $-\frac{3}{4}$, so the base of the small triangle in the lower right is $\frac{4}{3}$. Then

$$d = 5 - \frac{3}{4} - \frac{4}{3} - 2 = \frac{11}{12}.$$

10. Let $A$ and $B$ be $3 \times 3$ matrices with integer entries, such that $AB = A + B$. Find all possible values of $\det(A - I)$. Note: The symbol $I$ represents the $3 \times 3$ identity matrix.

**Solution:**

The given equation is equivalent to

$$AB - A - B + I = I \quad \text{or to} \quad (A - I)(B - I) = I.$$

Since the matrices have integer entries, the determinants of $A - I$ and $B - I$ are integers, and the last equation implies that $\det(A - I) = \det(B - I) = \pm 1$. Both cases are possible: if $A = B = O$, then $\det(A - I) = \det(-I) = -1$, and if $A = B = 2I$, then $\det(A - I) = \det(I) = 1$.